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What's the matter?

Our basic principle is the following:

- when we learn mathematics, we do not necessarily become professional mathematicians;
- conversely, if we want to learn mathematics effectively (develop our minds to understand what we do, keep in mind the essentials even if we don't use everyday), we must temporarily become a mathematician, and the classroom must act as a scientific community!

We claim that **if a student does not act mathematically, he cannot master mathematics,** because we consider that mathematics essentially reflect a particular view of reality, a specific way of thinking about the world: when a mathematician tries to understand a part of reality, when he wants to solve a specific problem, not only does he not stop dreaming, but rather he nurtures his dreams and simultaneous structures his imagination.
So he conjectures that he hopes, i.e. he takes a risk to tell himself and then to tell a community:
“I think that is true“ and immediately after he did that, to avoid taking dreams and desire in place of reality, he attempts to break what he hopes (that's not a natural attitude), he pushes himself to discover whether his conjecture is false, and only if he fails in that endeavor, does he try to prove that it is impossible for his idea to be false.

Finally the most important result of this specific way of thinking the world is this: a mathematician attempts to solve only conjecture. When his proof succeeds, he is naturally happy. But even if he discovers that his idea is false, unlike everyday life, he is still happy because he considers that his work is useful, not a waste of time. His work is necessary to discover what is misunderstood, what is “wrong or poor thinking”!

Scientific debate in mathematical course is a didactic engineering which tends to effectuate this principle, to implement this epistemological practice.

In fact, this practical structure lies on two other principles:

- *a socio-constructivist principle* : if an essential part of scientific knowledge is to give us the capacity to grasp situational complexity and act rationally, students must encounter and deal with the conflict between everyday rationality (common sense) and scientific rationality.
- *a socio-ethicist principle* : every human should have the opportunity to understand the deep meaning of what we strive to teach.

Ultimately, we realize that school is not just a place to obtain knowledge and diplomas, but also an important place to develop one's potential and acquire the habit of community life: the ability to understand other people's arguments, to create and develop one's own arguments, defend one's own thesis, even if the interlocutor is more knowledgeable, more powerful, older and wiser.
We believe that scholarly education may be a good means to discover the fecundity of common work, and that the mathematical approach, in particular, may be a powerful tool to acquire the autonomy essential for co-operation without domination.

In order to achieve this goal, we must establish a new didactic contract: students and teachers must consider examinations as a mean, a necessary tool for assessing competence, not as the central goal.

Principle of realization

During the course, students are considered as a mathematicians' community who must debate scientifically around the topics proposed by the teacher : they are invited to formulate conjectures and to take sides on their relevance and their truth.

What is radically different there compared to conventional education is the status of truth; for the student who honors this new contract, truth is not discovered exclusively in the professor's words or in books, but rather truth is to constructed together:

- first, the student must believe in what he conjectures,

- next, he must develop rational arguments to convince himself that his conjectures are not false,

- then he must find words, formulas, theorems and metaphors capable of persuading not only the professor as well as all his classmates, but also everyone else who understands mathematics.

To build truth from one's own assertion requires the ability to listen and understand the objections of classmates until, at last, one is able to disprove counter-arguments if they are incorrect. To build truth from a conjecture requires the ability to accept as fact, as knowledge, the discovery of one's own mistakes, and how one got there.

A consequence of this point of view is that in this debate, the professor should avoid revealing his own opinion about the truth of the subject under discussion - a very, very difficult challenge for a mathematics professor. Instead, he has to foster diversity of opinions with the objective of revealing conflicts in rationality.

For all these reasons, a teacher should:

- give sufficient time (neither too short nor too long - a delicate choice) for students to develop their own opinions based on scientific reasons, and record on the blackboard the global results of this individual work (for example: True 15, False 17, Other 40);

- write on the blackboard the arguments presented by students, without alteration, (also not an easy task), repeating aloud what may have been said too softly;

- strive to maximize the number of students involved in discovering a rational solution. The debate should be stopped only when a majority of students are well engaged by the problem.

When articulating syntheses and focusing on what is important to bear in mind, the professor should avoid congratulating whoever gave correct arguments, and avoid putting down those who gave incorrect solutions. Instead, he should seek to analyze the community 's erratic path and show how mistakes, once analyzed, can yield good ideas and useful solutions.

This is the same schema that I. Lakatos proposes in **Proofs and refutations** (but in a classroom, the actors are not professional mathematicians and the teacher must make an important memory work if he wants to maximize the participation of many students in the debate over a long time).

Difficulties and results

Of course this process is slow and requires a lot of investment from the professor, as well as agreement by students and professor to take risks: Will something interesting happen today ? Are we going to cover the syllabus ? Are we going to learn what we need to succeed later? etc. etc.

If this didactic practice is not an immediate success for everyone, (some students who in a first time sent out this practice are often very happy have met them a few years later), it has the capacity of transforming the progress of most students in their comprehension of scientific games and their own capacity to imagine the concept of truth and proof.

Here are certain observations by students after tree months of scientific debate, (I report these remarks not for publicity but to indicate the nature of change).

Students in the first year of University :

- This year I detect that it is possible to reflect in mathematics!

- In October math equalled a simple tool, holding no interest for me.
I have discovered that it is possible to take some pleasure in studying math.

- I never thought of mathematics as playing a role in reality; I viewed mathematics as a science entirely separate from reality. Modelisation has changed my point of view.

- I tell myself: "I want to speak", because if you want to speak, you have to be clear in your mind.

- I arise to amaze myself!

- Demonstration arises me as stick of prestidigitator, scientific debate shows that this sort of reasoning you can build yourself if you think about with method.

- I used to think that scientific debate was wasted time during which we learned too little, but I realize that we have learned much.

- What most annoyed me at first has now become most interesting: don't think of mathematics as a finite science, don't expect to learn miraculous formulas or to make impressive calculus.

Next, assorted remarks by mature students (in the last year before they become professor):

- What matters and is absent from traditional lesson mathematics is creativity, the possibility for everyone to be active, and not merely a passive spectator.

- We don't see so much knowledge but… in this way we can go deeper, we can ask ourselves because it is legitimate. In a conventional class, if the best questions seem to arise out of mathematical course, it is because we don't know if our own questions are good or absurd and the answers evident or not. The act of framing questions is a good way to reveal that you don't understand!

In scientific debate there is no reason to hide what you have not understood because you are not ashamed if you mistake.

- Scientific debate is both simple and very complicated. Simple, because what could be more natural than to engage in speculation which moves us to uncover new objects better adapted to our questions…; it is the way of research, however a complex quotidian practice!

What seems to me fundamental is this: at the base of democracy are three principles, liberty, equality, fraternity. The most forgotten is the third. Here, we succeed to remember it.

How is this process going to be worthwhile in the long run ?

Principally three sorts of situation of debate arise:

First sort : "Unexpected" debate

If one student asks: "May we write $f(A \cap B) = f(A) \cap f(B)$?"

Professor forbids himself to give the answer, but writes to the blackboard:

Conjecture " If f is a function, $f : E \rightarrow F$, and if A and B are two parts of E, then $f(A \cap B) = f(A) \cap f(B)$."

If after five minutes of individual reflection all students think the same thing and that this agreement is accurate, the professor decides to go on, but if (as is usually the case) opinions conflict, scientific debate becomes necessary to eliminate mistakes and construct a significant answer.

Second sort : situation built by professor to introduce a new concept

In first year of University, if you want to introduce the concept of integral as a sum which is able to adapt to the variation of a continuous parameter, you can choose the situation of bar.
(This situation is taken as an example in Michele Artigue's conference)

Question : How large is the attract strength F between this bar M and the punctual mass M'? (When m and m' are punctual masses at distance r, this attract strength f is $k.m.m'/r^2$.)

M = 18 kg M' = 2 kg

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……………………6 m……………… ……. 3 m …………

Students' answers after 15 ' of individual works:

F =	- 8 k	4/9 k	k	4/3 k	4 k	8 k	?
Students' nb	10	3	#50	8	10	4	#25

There, debate is very lively because majority of students think that it is possible to apply a false principle of center of gravity (result k).

Some students think then to cut this bar into two pieces and when they apply this principle, they find an other value (1,2k ≠ 1k) that is absurd if the principle of gravity center is true.
This idea can be taken by others students who maximize and minimize strength on every piece of bar, and so the debate naturally induces the idea of cutting more and more and maximizing and minimizing strength on every piece; the exact strength F arises then as a limit process.

We don't affirm that Riemann integral theory is built alone by this situation, we affirm only that with this sort of situation we are able to make sense of a complex mathematical procedure (cutting, maximizing, doing limit process) which is too abstract in itself for majority of students when it is introduced without adapted concrete problem.
(For further study see [1]& [4] and naturally also [7])

<u>Third situation : deepening of some concept or theory</u>

When the construction of integral is finished, if you want to work on integral of sup born or on continuity or on derivability, you can ask students to make conjectures on propriety of this integral.

<u>Working proposition</u>

If $F(x) = \int_a^x f(t)\,dt$, make some conjectures who link f properties to F properties.

After some time you obtain probably bad or false conjecture as :

- " f continuous F continuous", or " f increasing F increasing".

Solving this conjecture, good theorems arise as :

- " F is always continuous", " f ≥ 0 F increasing",

" f continuous F differentiable" etc. etc. //
(For further study, see [1] [2] [3] & [4].)

<u>Finally, what's essential with scientific debate</u> ?

Our goal is not to introduce the mystification: "you are able to discover alone all theorems in the domain of mathematical". Our goal is to show students, concretely, the need to act mathematically in order to understand mathematics: "your questions, your solutions, your arguments, bad or true, are necessary to build sense on abstract theories, are usefully to transcend natural naïveté, to build bridge between theory and practice, between mathematics and other sciences, between concepts and algorithms…. ".

The act of formulating a conjecture obliges his author to enter in a engage scientifically: "if I want assert this or that, is it reality ? do I deeply think this or that is true ? "

The act of collectively resolving conjectures shows us how it's usefully to be able to work with contradiction, to respect those who don't think as we do. If the professor is really neutral, every student can think: it is useful to have, to dare, to defend my ideas because different points of view can help everyone to better understand what is delicate.

What matters for a student is not to show what he knows, and hide what is unknown, but to do what helps everyone to better understand.

What matters for a professor is not to show that he knows everything and calculates very fast, but to show how he knows, how he discovers his own mistakes, how he succeeds to work on his own imagination and with the imagination of others.

The inevitable teacher's authority is now based on an another system of values, and the didactic contract must be explicit: scientific debate is possible if everybody respects the professor because he is competent, but if also this magisterial knowledge doesn't make students timid.

Didactic contract is accepted when everyone thinks that even if another student knows more or less or quicker or more complicate, those differences need not induce domination or defensiveness.

Finally, scientific debate requires con-fraternity to better understand and better explicate.

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