ISSN 1292-8763

International Newsletter on the Teaching and Learning of Mathematical Proof

La lettre de la Preuve
Automne 2001

On-line material

PUPA'S TWO COMPLEMENTARY PRODUCTS: TAXONOMY OF STUDENTS' EXISTING PROOF SCHEMES AND *DNR* -BASED INSTRUCTION

Guershon Harel
University of California, San Diego

PUPA'S TWO COMPLEMENTARY PRODUCTS: TAXONOMY OF STUDENTS' EXISTING PROOF SCHEMES AND *DNR*-BASED INSTRUCTION[1]

Guershon Harel
University of California, San Diego
harel@math.ucsd.edu

PUPA is an NSF-funded research project[2] that studied the development of students' **p**roof **u**nderstanding, **p**roduction, and **a**ppreciation. PUPA's central research questions include:

- What are students' conceptions of proof?
- What sorts of experiences seem effective in shaping students' conception of proof?
- Are there promising frameworks for teaching the concept of proof so that students appreciate the value of justifying, the role of proof as a convincing argument, the need for rigor, and the possible insights gained from proof?

PUPA's research subjects were mathematics majors, including preservice secondary mathematics teachers, and engineering majors. Two of the main products produced by PUPA are:

(a) An extensive taxonomy of students' existing proof schemes.
(b) A system of pedagogical principles, called *DNR*—an acronym for the three leading principles in the system—*duality*, *necessity*, and *repeated-reasoning*—aimed at enhancing students' conception of mathematics in general and that of proof in particular.

The two products complement each other in that while the former aimed at mapping students' existing conceptions of proof, the latter specifies foundational pedagogical principles for enhancing students' proof schemes. A bird's eye view of the taxonomy of proof schemes is in Figure 1 below. For a full description of the taxonomy and of cognitive, historical and philosophical analyses on which the taxonomy is based, see Harel & Sowder (1998), and Harel (1999, 2001, In Press).

The second product—the *DNR*-based instructional treatment—is the focus of this brief report. PUPA's teaching experiments show that through *DNR*-based instruction students gradually refine their existing proof schemes into *transformational proof schemes*—mathematically mature ways of thinking, where students formulate general conjectures and use logical deduction to reach conclusions. Further, together with this conceptual change in their ways of thinking about proof, students develop solid understanding of the respective topic taught.

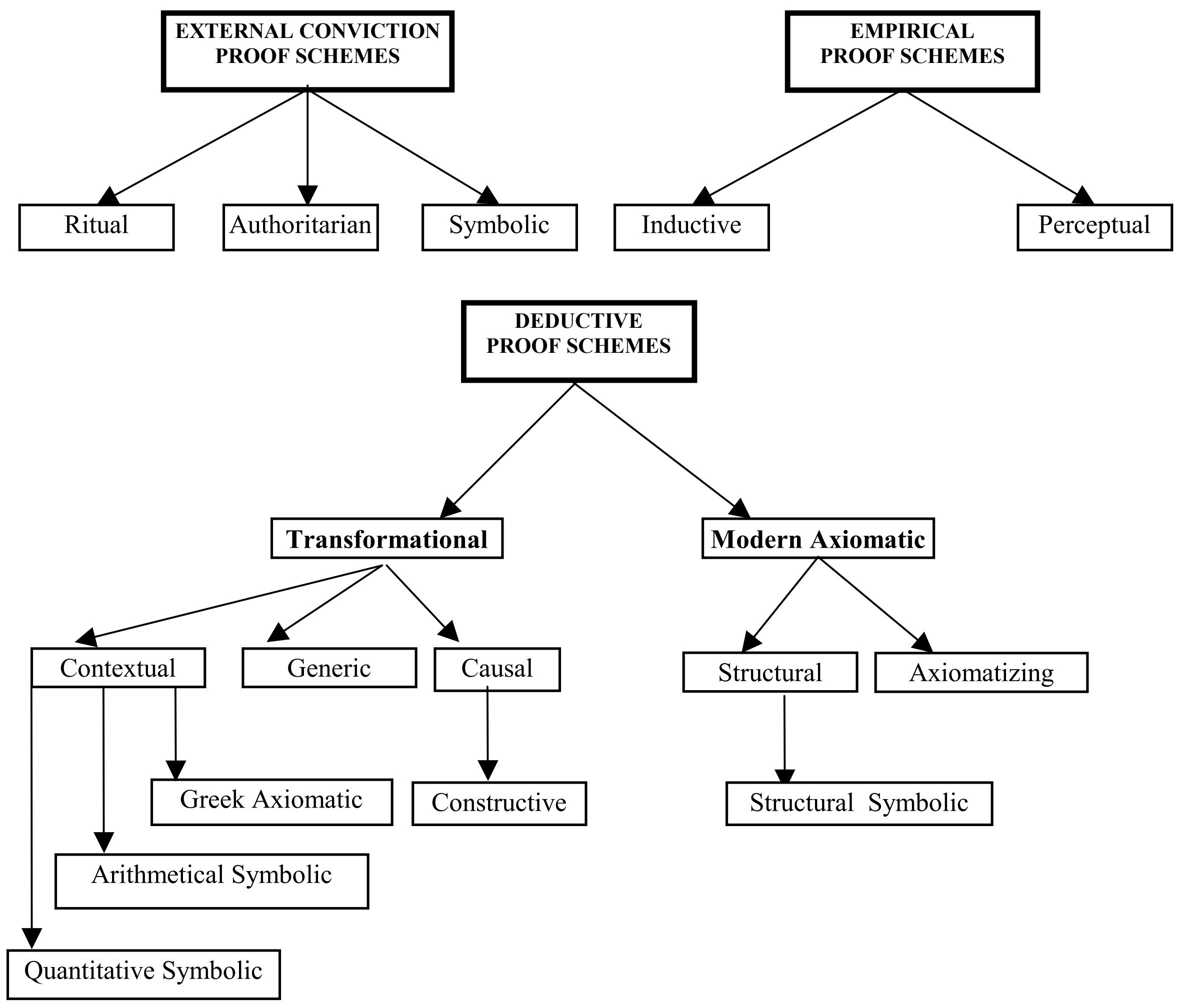
Encouraged by the success of *DNR*-based instruction in enhancing undergraduate students' conceptions of proof, we have conducted pilot experiments to study its effectiveness in professional development courses for inservice junior- and senior-high-school algebra teachers. Current observations suggest that *DNR*-based instruction has brought about a significant change

[1] Many of the ideas presented in this report appeared in Harel (1998, 2000, 2001, In Press) and Harel & Sowder (1998, Submitted).

[2] The research presented here is part of the PUPA (Proof Understanding, Production, and Appreciation) Project, supported, in part, by a grant from the National Science Foundation to Larry Sowder of San Diego State University and the author of this paper. Opinions expressed are those of the author and not necessarily those of the Foundation.

in teachers' knowledge of algebra, in their use of mathematical justification, in their understanding of how students learn, and in their practices of teaching mathematics. The goal of this short report is to give a synopsis of the *DNR*-based instruction through its three foundational principles: the ***duality principle***, the ***necessity principle***, and the ***repeated reasoning principle***. For more information, see Harel (1998, 2000, 2001) and Harel & Sowder (submitted).

Figure 1: A Taxonomy of Students' Existing Proof Schemes



The Duality Principle

Fundamental to *DNR*-based instruction is the distinction between *ways of thinking* (WoT) and *ways of understanding* (WoU) (Harel, 1998). The phrase *way of understanding* refers to the specific mathematical actions one takes in doing mathematics. Such actions can be one of three categories:

1. The particular meaning/interpretation one gives to a concept, relationship between concepts, statement, or problem.
2. The particular solution one provides to a problem.

3. The particular evidence a person offers to establish or refute a mathematical assertion.

WoU Category 1*: *Particular Meaning/Interpretation

Consider the concept of fraction. A student may understand this concept in terms of *unit fraction* (a/b is a $1/b$ units); in terms of *part-whole* (a/b is a units out of b units); in terms of *partitive division* (a/b is the quantity that results from a units being divided equally into b parts); in terms of *quotitive division* (a/b is the measure of a in terms of $b-$units). All of these would be ways of understanding fractions.

Similarly, the concept of *scalar equation* can be understood, for example, in terms of a condition, in terms of a real-valued function, or in terms of a proposition-valued function. One can understand the string of symbols, $y=\sqrt{6x-5}$, as a condition on the variables x and y (i.e., "the quantity y equals the square root of the quantity, $6x-5$") or as the real-valued function $y(x)=\sqrt{6x-5}$ ("For an input x there correspond the out put $\sqrt{6x-5}$"), or as the proposition-valued function over the set of all ordered pairs (x,y) or real numbers:

$$E((x,y))=\begin{cases} True & \text{if} \quad y=\sqrt{6x-5} \\ False & \text{if} \quad y\neq\sqrt{6x-5} \end{cases}.$$

These, of course, are examples of mature ways of understanding the concept of scalar equation. They can be markedly different from how some students' understand this concept. For example, Kieren (1992) reports that Behr, Erlwanger, & Nichols (1976) found that beginning algebra students understand the equal sign as a "do something signal," where one side of the equation is reserved for the operation to be carried out and the other side for its outcome. Students with this way of understanding accept a statement such as $4+3=7$ but are reluctant to accept a statement such as $4+3=6+1$ or $3=3$, for while in the first equation the equal sign separates the operation $(4+3)$ from its result (7), in the second and third equations the expressions on the two sides of the equal sign both are operations ($4+3$ and $6+1$) or both are entities (3).

WoU Category 2: Particular Solution

Examples of ways of understanding from the second category–particular methods of solving a problem–can be seen in the following. A ninth-grade class was assigned the problem

> A pool is connected to two pipes. One pipe can fill the pool in 20 hours, and the other in 30 hours. How long will it take the two pipes together to fill the pool?

Among the different solutions provided by the students in the class, there were the following four—each represents a different way of understanding.

> ***Solution 1***: Divide the pool into five equal parts. The first pipe would fill one part in four hour, and the second pipe in six hours. Hence, in 12 hours the first pipe would fill $3/5$ of the pool and the second pipe the remaining $2/5$.
>
> The student who provided this solution accompanied it with a sketch similar to following sketch.

6	6	4	4	4

Solution 2: It will take the two pipes 50 hours to fill the pool.

Solution 3: It will take the two pipes 10 hours to fill the pool.

Solution 4: It would take x hours. In one hour, the first pipe will fill $1/20$ of the pool whereas the second will fill $1/30$. In x hours the first pipe would fill $x/20$ and the second, $x/30$. Thus, $x/20 + x/30 = 1$. (The student then solved this equation to obtain $x = 12$.)

WoU Category 3: Particular Justification

Examples of ways of understanding for the third category—justifying or proving—include the following responses by prospective secondary teachers to the problem:

Prove that $\log(a_1 \cdot a_2 \cdot \cdots \cdot a_n) = \log a_1 + \log a_2 + \cdots + \log a_n$ for all positive integers n.

Justification 1:

$\log(4 \cdot 3 \cdot 7) = \log 84 = 1.924$

$\log 4 + \log 3 + \log 7 = 1.924$

$\log(4 \cdot 3 \cdot 6) = \log 72 = 1.857$

$\log 4 + \log 3 + \log 6 = 1.857$

Since these work, then $\log(a_1 \cdot a_2 \cdot \cdots \cdot a_n) = \log a_1 + \log a_2 + \cdots + \log a_n$.

Justification 2:

$\log(a_1 a_2) = \log a_1 + \log a_2$ by definition

$\log(a_1 a_2 a_3) = \log a_1 + \log a_2 a_3$. Similar to $\log(ax)$ as in step (i), where this time $x = a_2 a_3$.

Then

$\log(a_1 a_2 a_3) = \log a_1 + \log a_2 + \log a_3$

We can see from step (ii) any $\log(a_1 a_2 a_3 \cdots a_n)$ can be repeatedly broken down to

$\log a_1 + \log a_2 + \cdots + \log a_n$.

In our usage, then, the phrase *way of understanding* conveys the reasoning one applies in a local, particular mathematical situation. The phrase *way of thinking* on the other hand refers to what governs one's ways of understanding, and thus expresses reasoning that is not specific to one particular situation but to a multitude of situations. A person's way of thinking (***WoT)*** involves three interrelated categories of knowledge: ***beliefs***, ***problem-solving approaches***, and ***proof schemes***.

WoT Category 1: Beliefs—Views of Mathematics

"Formal mathematics has little or nothing to do with real thinking or problem solving," and "The solution of a problem should not take more than five minutes" are common beliefs among students (Schoenfeld, 1985, p. 43). On the other hand, in our work with undergraduate mathematics students we found beliefs such as "A concept can have multiple interpretations" and "It is advantageous to possess multiple interpretations of a concept," although essential in courses such as linear algebra, are often absent from the students' repertoires of reasoning. The development of these ways of thinking should not wait until students take advanced-mathematics courses, such as linear algebra; rather, students must begin acquiring in early age. Indeed elementary mathematics and secondary mathematics are rich with opportunities for the students to develop these ways of thinking. For example, the different ways of understanding fractions presented above should provide such an opportunity for all elementary-grade students. Students should learn, for example, that the fraction 3/4 can be understood in different ways: 3 individual objects, each of quantity 1/4; the result when 3 objects are shared among 4 individuals; the portion of the quantity 4 that equals the quantity 3; and 3/4 as a mathematical object, a conceptual entity, a number. In turn, these multiple ways of understanding are likely to enable students vary the meaning of operations on fractions according to the context of the problem in hand. For example, while $3 \div (1/4)$ is amenable to quotitive interpretation, $(1/4) \div 3$ is amenable to partitive interpretation.

WoT Category 2: Problem-solving approaches[3]

"Look for a simpler problem," "Consider alternative possibilities while attempting to solve a problem," "Look for a key word in the problem statement" are example of problem-solving approaches. The latter way of thinking might have governed the way of understanding expressed in Solution 2 to the Pool Problem. What ways of thinking might have governed the other three solutions to this problem? Of particular interest is Solution 1. Only one student, G, provided this solution. When interviewed, G indicated that she drew a diagram—a rectangle to represent the pool (see the rectangular sketch above)—and divided it into 5 equal parts. Then she noticed that $3(20/5)$ is the same as $2(30/5)$. G was unwilling (or unable) to answer the question of how she thought to divide the rectangle into 5 equal parts, and so we can only conjecture that a juxtaposition of ways of thinking had driven G's solution. They may include "Draw a diagram," "Guess and check," and "Look for relevant relationships among the given quantities." It was shocking to learn that G's score on this problem, as well as on three other problems she correctly solved in a similar manner (i.e., without any "algebraic representation") was zero. Her teacher's justification for this score was something to the effect that G did not solve the problems algebraically, with unknown and equations, as she had been instructed to do. And, he continued to argue, it is only by following his instructions would G learn algebra.

WoT Category 3: Proof schemes

One of the most ubiquitous proof schemes held by students is the *inductive proof scheme*, where students ascertain for themselves and persuade others about the truth of a conjecture by direct measurements of quantities, numerical computations, substitutions of specific numbers in algebraic expressions, etc. This way of thinking governs the way of understanding expressed in Justification 1. The way of understanding expressed in Justification 2, on the other hand, is a manifestation of a different way of thinking, called *transformational proof scheme*. In Harel

[3] We chose not to use the term "heuristics" here because while every heuristic is a general approach to solving problems—and hence a way of thinking, the converse is not true. Heuristics are defined as "rules of thumb for effective problem solving" (Schoenfeld, 1995, p. 23); students' approaches to solving mathematical problems, needless to say, are not always heuristics in this sense.

(2001) it is shown why Justification 2 contains the three essential elements that characterize the transformational proof scheme: (a) consideration of the generality aspects of the conjecture, (b) application of mental operations that are goal oriented and anticipatory—an attempt to predict outcomes on the basis of general principles—and (c) transformations of images that govern the deduction in the evidencing process.

Related to the notion of way of thinking is the phrase *habit of mind* (Cuoco, Goldenberg, & Marks, 1996). In our usage, habits of mind are qualities of ways of thinking; namely, a habit of mind is an *internalized* way of thinking—one applies it autonomously, and often spontaneously.

The distinction between ways of thinking and ways of understanding is entailed from the fundamental premise that one's specific actions are induced and governed by her or his general views of the world, and, conversely, one's general views of the world are formed by her or his specific actions; hence, the Duality Principle:

> ***Students' ways of thinking impact their ways of understanding mathematical concepts. Conversely, how students come to understand mathematical content influences their ways of thinking.***

Students' current ways of thinking—according to this principle—are formed to a large extent by their practice in communicating mathematical ideas, in solving mathematical problems, and in justifying mathematical assertions. Conversely, students' ways of thinking are a major cause for the ways of understanding manifested in their mathematical behaviors.

Further, and of particular importance, the future development of students' ways of thinking depends critically on the nature of the specific mathematical activities in which they are currently engaged. Specifically, students would be able to modify, refine, or advance their current ways of thinking *only* through appropriate ways of understanding in (a) communicating mathematics, (b) solving mathematical problems, and (c) justifying and proving mathematical assertions.

Still further, ways of thinking cannot be taught directly (e.g., it is of little or no gain telling students that proofs in geometry cannot rely on measurement, or that algebra is important to their future careers). Ways of thinking can only be extracted and abstracted by the students themselves from meaningful problem-solving activities.

Current textbooks and teaching practices do not pay enough attention to ways of thinking. For example, seldom do we consider questions such as: What are the specific ways of thinking that common instructional activities in algebra—such as simplifying, factoring, equation solving, etc.—aim at promoting? Or, what exactly are the ways of thinking that can or should be promoted by the use of computer technology in algebra?

The immediate implication of the Duality Principle is that it is essential that teachers (a) form instructional goals in terms of ways of thinking and (b) devise and use appropriate instructional activities through which students can build ways of understanding that can potentially lead to the construction of desirable ways of thinking. Of course, two critical questions are yet to be answers: What constitute such appropriate instructional activities? What is the nature of instructional treatments that can help students construct desirable ways of understanding and ways of thinking? These questions fall in the territory of the other two principles: the *necessity principle* and the *repeated reasoning principle*.

The Necessity Principle

Fundamental to the *DNR*-based instruction is the premise that problem solving is not just a *goal* but also the *means* for learning mathematics. "Problem solving" is usually defined as "[engagement] in a task for which the solution method is not known in advance" (PSSM, 2000). Many of the situations students encounter in school satisfy this definition and yet they do not constitute problem solving because the situations are not *intrinsic*, but *alien*, to the students. Before we can formulate the *necessity principle,* the last two terms must be defined:

When we talk about mathematics problems in the context of education, we refer to a learning-teaching event that involves two sets of interpretations: the set that belongs to the poser of the problem and the set that belongs to the one to whom the problem is posed. The importance of this simple observation is in the fact that the two sets are not always congruent, and in many cases they do not even intersect. Teachers, however, are not always aware of this fact. They might present a problem to their class and incorrectly assume that their students share their interpretation(s) of the problem. Conversely, students might pose a question to their teacher, who either encounters difficulty understanding what they are asking or interprets their question differently from what it is intended. A mathematics problem can, therefore, be viewed as a triple consisting of: (a) the problem statement, (b) the set of the student's interpretations of the problem, and (c) the set of the teacher's interpretations of the problem.

Of particular interest are the scenarios where the two sets do not intersect because what is conceived as a problem by the teacher is unproblematic to the students, and vice versa. In general, a situation that is problematic to one person but is unproblematic to another is referred to as *intrinsic* (I) to the first and *alien* (A) to the second. Accordingly, mathematics problems in a learning-teaching setting are of four categories: (S,A,I), (S,I,A), (S,I,I), and (S,A,A).[4] (S,I,I)—the situation where a problem statement is intrinsic to both the student and the teacher—is the only desired category among the four. As long as the problem is intrinsic to both significant learning is likely to occur, even if, or perhaps especially when, the teacher's interpretations are different from those of the student.

Unfortunately, neither one of the other three cases, (S,A,I), (S,I,A), and (S,A,A), is uncommon. The latter, for example, occurs especially when the teacher lacks basic understanding of the mathematical content he or she is teaching—a not uncommon phenomenon as it has been widely documented. To illustrate, consider the following episode: A ninth-grade teacher demands his class to accompany each assertion written on the left-hand side of two-column proofs by a "justification" on the right-hand column. In one of the proofs written by a student in this class, she justified the assertions, "$AB \cong AB$" and "If $\sphericalangle ABC = 30^0$ and $\sphericalangle CBD = 45^0$, then $\sphericalangle ABC = 75^0$," by the phrases "reflexive property" and "additive property," respectively. It turned out that neither the student nor the teacher understood the meaning of these phrases, and both viewed the two assertions to be self-evident—ones that require no justification. Thus, the task to justify these assertions was alien to both the teacher and his student—a clear (S,A,A) case. On the other hand, if this teacher possessed the axiomatic proof scheme, he would have understood the meaning and role of the reflexive and additive properties in Euclidean geometry, and the situation would have been (S,A,I)—intrinsic to the teacher but likely alien to a ninth-grade student.

---

[4] The first two categories are different in that while (S,A,I) represents the case where the problem is alien to the student and intrinsic to the teacher, (S,I,A) represents the reverse case—the problem is intrinsic to the student and alien to the teacher.

Teachers' demand to solve problems by explicit algebraic symbols, such as equations—as we have seen earlier with G's teacher—is another example of (S, A, I). G's solution to the Pool Problem and her teacher's evaluation of the solution are a case in point: G probably did not see a need to represent the problem situation by an equation, for she was able to arrive at the solution to the problem by utilizing her existing conceptual tools.

Cases of the (S, A, I) category are the most common in mathematics education in all levels, but especially in the college level, where the teachers have deep understanding of the problems they present to their students but usually are unaware that their interpretations and appreciation of the problems are not always shared by their students. It is enough to randomly pick a standard elementary undergraduate calculus textbook to witness this phenomenon: theorems are stated with conditions whose need is entirely alien to the students (e.g., If f is continuously differentiable in an open interval (a,b), then …) The necessity of such conditions is seldom addressed.

Thus, the *necessity principle* asserts:

> ***For students to learn they must see the need of what we intend to teach them, where by 'need' it is meant intellectual need as opposed to social or economic need.***

The term *intellectual need* refers to a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate. For example, students might encounter a situation that is incompatible with, or presents a problem that is unsolvable by, their existing knowledge. Such an encounter is *intrinsic* to the learners for it stimulates a desire with them to search for a resolution or a solution, whereby they might construct new knowledge. There is no guarantee that the learners construct the knowledge sought or any knowledge at all. But whatever knowledge they construct, it is meaningful to them in that it is integrated within their existing cognitive schemes because it is a product of effort that stems from and is driven by their personal, intellectual need.

Whereas one should not underestimate the importance of students' *social need* (e.g., mathematical knowledge can endow me with a respectable status in my society) and *economic needs* (e.g., mathematical knowledge can help me obtain comfortable means of living) as learning factors, teachers should not and cannot be expected to stimulate, let alone fulfill, these needs. Intellectual need, on the other hand, is prime responsibility of teachers and curriculum developers.

The implementation of the *necessity principle* involves (a) recognizing what constitutes an intellectual need for a particular population of students relative to the concept to be learned, (b) developing a system of problem situations that correspond to their intellectual need, and from whose solutions the concept can be formed, internalized, or interiorized, and (c) creating an instructional environment in which the student can elicit the concept through engagement with the system. These are not three steps of a recipe to be carried out chronologically. Rather, these constitute, respectively, the essence of three inseparable aspects of mathematics education research—research in learning, curriculum development, and teaching.

The following two examples demonstrate the general idea (definitely not the complete instructional plan) of these steps in implementing the *necessity principle*. The first example demonstrates the creation of an intellectual need for computation, the second the creation of an intellectual need for communication:

Example 1: To help students construct ways of understanding the pivotal cluster of the concepts, “linear combination,” “dependence,”, and “independence,” we utilized one of the historical roots of linear algebra: systems of linear equations:

Students entering their first course in linear algebra are familiar with system of equations and understand their importance (in solving word problems, for example). They can be brought—as our experience suggests—to appreciate the fact that in some cases one cannot or does not want to solve a given system $AX = b$, and yet he or she needs to determine whether it has a solution or whether its solution is unique. We pose these Existence and Uniqueness problems early in our matrix-oriented course—before we mention any of the above concepts—to focus students' attention on a definite goal. Because we strongly emphasize—in fact define—matrix multiplication via the relation $N^{(k)} = \sum_j (N^{(k)})_j M^{(j)}$ (where $N^{(k)}$ represent the k-th column of the matrix N), it is not uncommon that a few students give a correct response to the Existence problem; namely, for $AX = b$ to have a solution b must be expressed as $x_1 A^{(1)} + x_2 A^{(2)} + \cdots + x_n A^{(n)}$. Of course, students seldom give this clean answer, but with the teacher's help, the class as a whole understands why the suggested relation between b and the columns of A merits attention and, therefore, deserves a name—“linear combination.”

Building on this students' understanding, we help students elicit the concepts of “dependence” and “independence” from the solution to the Uniqueness problem. To avoid unnecessary complications—something we learned from experience—we first pose this problem with a homogeneous system $AX = 0$. Students are now prepared to see that the relations “one of the columns of A is a linear combination of the other columns” and its negation solve the Uniqueness problem. Once this is achieved, new concepts are born, and names are designated to them: “linear dependence” and “linear independence", respectively.

***Example 2*:** In one of our teaching experiments in geometry, we introduced a hypothetical participant, who was called Ms. Smart. This participant is an intellectually able creature with whom the students can communicate in their own natural language, including the language of basic set-theory, but possesses none of the physical senses, such as visual and tactile perceptions. The class is presented with the task of communicating to Ms. Smart geometric concepts, conjectures, propositions, and justifications they have formed intuitively and transformationally. The idea is to bring the students to realize that in order to facilitate such a communication, they must formulate certain “agreements” with Ms. Smart. These agreements amount to a system of axioms. In the beginning, the instructor played the role of Ms. Smart, but gradually the students took on her or his persona.

The Repeated Reasoning Principle

Research has shown that repeated experience, or practice, is a critical factor in enhancing, organizing, and abstracting knowledge (Cooper, 1991). *The question is not whether students need to remember facts and master procedures but how they should come to know facts and procedures and how they should practice them*. This is the basis for the Repeated Reasoning Principle:

> ***Students must practice reasoning in order to internalize and interiorize ways of thinking and ways of understanding.***

Consider the following two important ways of thinking: "mathematical efficiency" and "transformational reasoning."

Two elementary school children, S and T, were taught division of fractions. S was taught in a typical method, where he was presented with the rule $(a/b) \div (c/d) = (a/b) \cdot (d/c)$, and the rule was introduced to him in a meaningful context and with an adequate justification that he understood. T, on the other hand, was presented with no rule. Each time she encountered a division of fraction problem, she explained its meaning and the rationale of her solution. S and T were assigned homework problems on division of fractions. S solved all the problems correctly, and gained, as a result, a good mastery of the division rule. It took T a much longer time to do her homework. Here is what T—a real third-grader—said when she worked on $(4/5) \div (2/3)$:

> How many $2/3$s in $4/5$? I need to find what goes into both [meaning: a unit-fraction that divides $4/5$ and $2/3$ with no remainders]. $1/15$ goes into both. It goes 3 times into $1/5$ and 5 times into $1/3$, so it would go 12 times into $4/5$ and 10 times into $2/3$ (She writes: $4/5 = 12/15$; $2/3 = 10/15$; $(4/5) \div (2/3) = (12/15) \div (10/15)$. How many times does $10/15$ go into $12/15$? How many times do 10 things go into 12 things? One time and $2/10$ of a time, which is 1 and $1/5$.

T had opportunities for reasoning of which S was deprived. T practiced reasoning and computation; S practiced only computation. Further, T eventually discovered the division rule and learned an important lesson about mathematical efficiency—a way of thinking S had little chance to acquire.

A key to the concept of mathematical proof is the transformational proof scheme–a scheme characterized by considerations of the generality aspects of the conjecture, application of mental operations that are goal oriented and anticipatory, and transformations of images as part of a deduction process. Educating students toward ways of thinking that help them build and reinforce transformational proof schemes must not start in college. Otherwise, years of instruction that focus on the results in mathematics, rather than the reasons behind those results, can leave the impression that only the results are important in mathematics, an opinion sometimes voiced even by mathematics majors.

The emphasis of *DNR*-based instruction is *repeated reasoning* that reinforces desirable ways of understanding and ways of thinking. Repeated reasoning, not mere drill and practice of routine problems, is important for internalizing and consolidating concepts. The sequence of problems must continually call for thinking through the situations and solutions; thus, they must respond to the students' changing intellectual needs.

DNR as a System

The three principles, *duality*, *necessity*, and *repeated reasoning*, constitute a system in the sense that they interdependently address three fundamental aspects of the learning-teaching process.

- ***Instructional objectives***. In designing, developing, and implementing mathematics curricula, ways of thinking and ways of understanding must be the ultimate cognitive objectives. They must be addressed simultaneously, for each affects the other.
- ***Formation of Concepts***. Meaningful concepts can only be elicited through solutions to problems that correspond to students' intellectual needs. The concepts elicited, as well as

the ways in which they are elicited, constitute and at the same time impact students' ways of thinking and ways of understanding.

- ***Internalization/Consolidation of concepts***. Ways of thinking and ways of understanding are internalized and solidified through repeated solutions to problems students understand and appreciate.

References

Behr, Erlwanger, & Nichols (1976). *How Children View Equality Sentences* (PMDC Technical Report No. 3). Tallahassee: Florida State University. (ERIC Document Reproduction Service No. ED144802.)

Cooper (1991). The role of mathematics transformation and practice in mathematical development. In L. P. Steffe (Ed.), *Epistemological foundations in mathematical experience*, pp. 102-123. New York: Springer-Verlag.

Cuoco, A., Goldenberg, E. P, & Marks, J. (1996). Habits of mind: an ongoing principle for mathematics curricula. *Journal of Mathematical Behavior, 15*, 375-402.

Harel, G. (1998). Two Dual Assertions: The First on Learning and the Second on Teaching (Or Vice Versa). *The American Mathematical Monthly*, 105, 497-507.

Harel, G., & Sowder, L. (1998). Students' proof schemes. *Research on Collegiate Mathematics Education*, Vol. III. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), AMS, 234-283.

Harel, G. (1999). Students' understanding of proofs: a historical analysis and implications for the teaching of geometry and linear algebra, *Linear Algebra and Its Applications, 302-303*, 601-613.

Harel, G. (2000). Three principles of learning and teaching mathematics: Particular reference to linear algebra—Old and new observations. In Jean-Luc Dorier (Ed.), *On the Teaching of Linear Algebra,* Kluwer Academic Publishers (pp. 177-190).

Harel, G. (In Press). Students' Proof Schemes Revisited: Historical and Epistemological Considerations. In P. Boero (Ed.), *Theorems in School*, Kluwer.

Harel, G. (2001). The Development of Mathematical Induction as a Proof Scheme: A Model for *DNR*-Based Instruction. *Journal of Mathematical Behavior*.

Harel, G., & Sowder, L. (Submitted). Advanced Mathematical-Thinking: Its Nature and Its Development.

Kieren (1992). The learning and Teaching of School Algebra. In D. Grouws (Ed). *Handbook for Research on Mathematics Teaching and Learning*. New York: Macmillan, 296-333

PSSM (2000). Principles and Standards for School Mathematics. Reston, VA: National Council of Teachers of Mathematics, Inc.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Harcourt Brace Jovanovich.