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Counter-Examples in Teaching/Learning of Calculus: Students' Performance

Sergiy Klymchuk

Auckland University of Technology
Auckland, New Zealand

ABSTRACT: This paper presents a case study that involved first year science and engineering students at the Auckland University of Technology, New Zealand. After very positive feedback was received from students in the international study on their attitudes towards counter-examples in the teaching/learning of Calculus (Gruenwald, Klymchuk, 2003) it was decided to investigate how the use of counter-examples would affect students' performance. The case study showed that the usage of counter-examples significantly improved students' performance on a test question that required conceptual understanding, but did not affect their performance on other test questions that only required the application of familiar rules, algorithms and calculations.

INTRODUCTION

Counter-examples provide an important means of communicating ideas in mathematics, whose entire history may be viewed as making conjectures and then either proving or disproving them by counter-example. Here are a few well-known cases to illustrate the point:

1. For a long time mathematicians tried to find a formula for prime numbers. The numbers of the form $2^{2^n}+1$, where n is natural were once considered as prime numbers, until a counter-example was found. For n = 5 that number is composite: $2^{2^5}+1=641\times 6700417$.

2. Another conjecture about prime numbers is still waiting to be proved or disproved - Goldbach's or the Goldbach-Euler conjecture, posed by Goldbach in his letter to Euler in 1742. It looks deceptively simple at first. It states that *every even number greater than 2 is the sum of 2 prime numbers*. For example, 12 = 5 + 7, 20 = 3 + 17, and so on. A powerful computer was used in 1999 to search for counter-examples to that conjecture. No counter-examples have been found up to 4×10^{14}. In 2000 the book publishing company Faber & Faber offered a US$1 million prize to anyone who could prove or disprove that conjecture. To date (April, 2005) the prize remains unclaimed.

3. In the 19th century the great German mathematician Weierstrass constructed his famous counter-example – the first known fractal – to the statement: *a function continuous on (a,b) cannot be non-differentiable at any point on (a,b)*. Many mathematicians at that time thought that such 'monster-functions' that were continuous but not differentiable at any point were absolutely useless for practical applications. About a hundred years later Norbert Winer, the founder of cybernetics pointed out in his book "I am a mathematician" that such curves exist in nature – for example, they are trajectories of particles in Brownian motion. In recent decades such curves have been investigated in the theory of fractals – a fast growing area with many applications.

Using counter-examples in teaching/learning of Calculus can be beneficial in many areas:

- For deeper conceptual understanding
- To reduce or eliminate common misconceptions
- To advance one's mathematical thinking, that is neither algorithmic nor procedural

- To enhance generic critical thinking skills – analysing, justifying, verifying, checking, proving which can benefit students in other areas of life
- To expand the ‘example set’ - a number of examples of interesting functions for better communication of ideas in mathematics and in practical applications
- To make learning more active and creative

The international study on students’ attitudes towards the usage of counter-examples as a pedagogical strategy in the teaching/learning of Calculus (Gruenwald, Klymchuk, 2003) involving over 600 students from 10 universities around the world showed students’ attitudes to be very positive. 92% of the participants reported that this strategy was very effective. Many of them commented that it helped them to understand concepts better, prevent mistakes, develop logical and critical thinking skills, and that it made their participation in lectures more active. Students’ attitudes and their exam performance are different matters though, so this study investigates how the use of counter-examples affects students performance.

THE STUDY

Two groups of students enrolled in science or engineering courses from the Auckland University of Technology were selected for this case study. In group A there were 14 students and in group B (the control group) there were 11. All of the students had similar mathematics backgrounds and ages, and all were Chinese. Both groups attended 3 lectures and 1 tutorial per week with the same lecturer, except group A was taught by another lecturer once a week who spent 5-6 minutes (out of a 50 minute lecture) on counter-examples. There were 8 weeks before the mid-semester test, so counter-examples were used in a total of 8 lectures. During this 8-week period the entire time spent on counter-examples in the lectures was about 45 minutes.

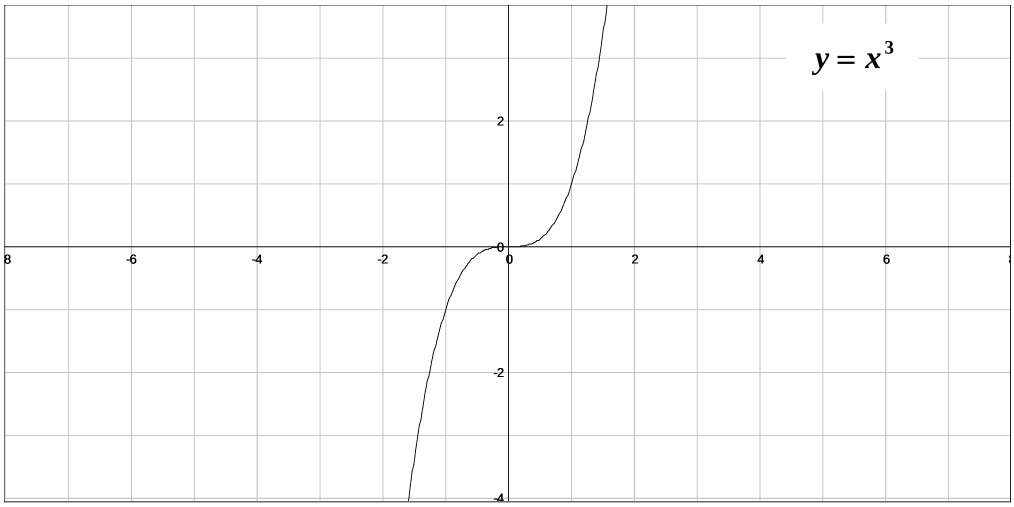
Below are some statements and related counter-examples that were discussed in the group A’s lectures.

Statement 1

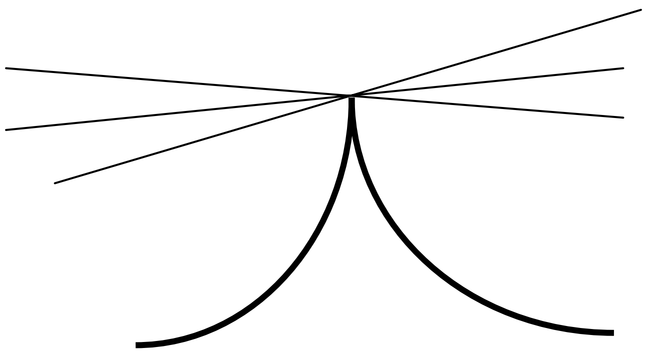
The tangent to a curve at a point is the line which touches the curve at that point but does not cross it there.

Counter-example

a) The x-axis is the tangent line to the curve $y = x^3$ but it crosses the curve at the origin.



b) The three straight lines just touch but don't cross the curve below at the peak, but none of them is the tangent line to the curve at that point.



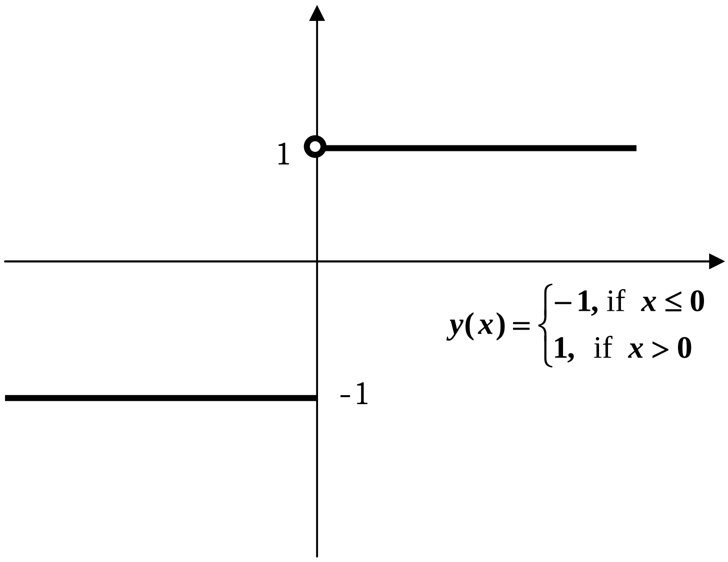
Statement 2

If the absolute value of the function y = f(x) is continuous on (a,b) then the function is also continuous on (a,b).

Counter-example

The absolute value of the function

$y(x)=\begin{cases}-1, \text{ if } x\le 0\\ 1, \text{ if } x>0\end{cases}$ is $|y(x)|=1$ for all real x and it is continuous, but the function y(x) is discontinuous.

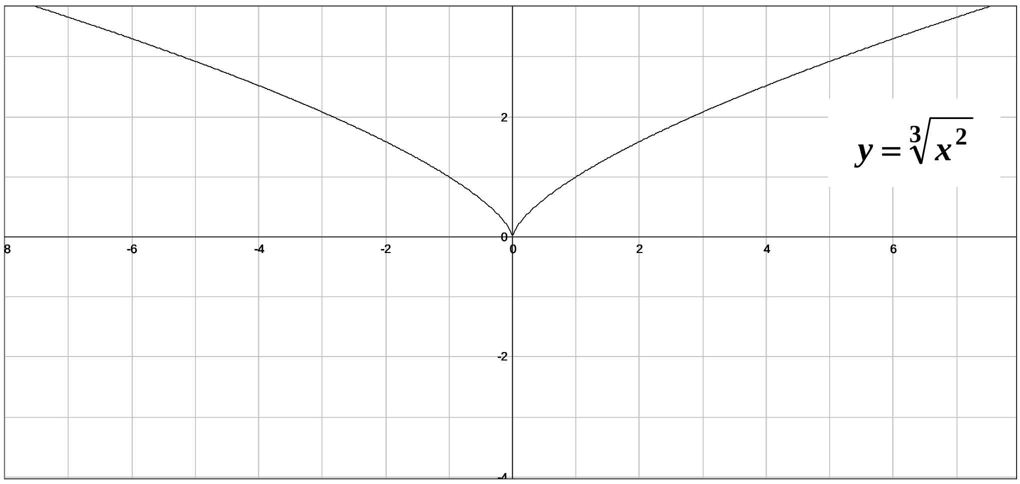


Statement 3

If a function is continuous on R and the tangent line exists at any point on its graph then the function is differentiable at any point on R.

Counter-example

The function $y=\sqrt[3]{x^2}$ is continuous on R and the tangent line exists at any point on its graph but the function is not differentiable at the point x = 0.

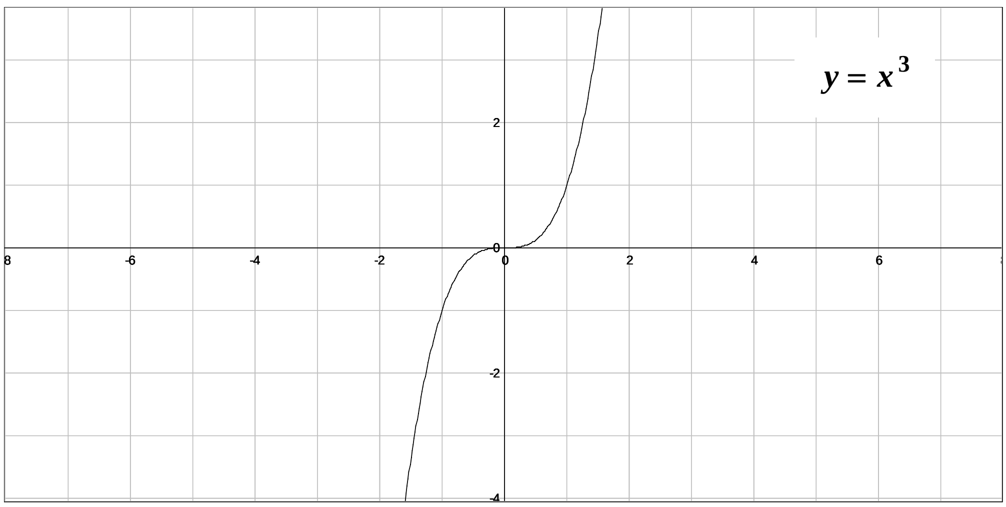


Statement 4

If the derivative of a function is zero at a point then the function is neither increasing nor decreasing at this point.

Counter-example

The derivative of the function $y = x^3$ is zero at the point x = 0 but the function is increasing at this point.

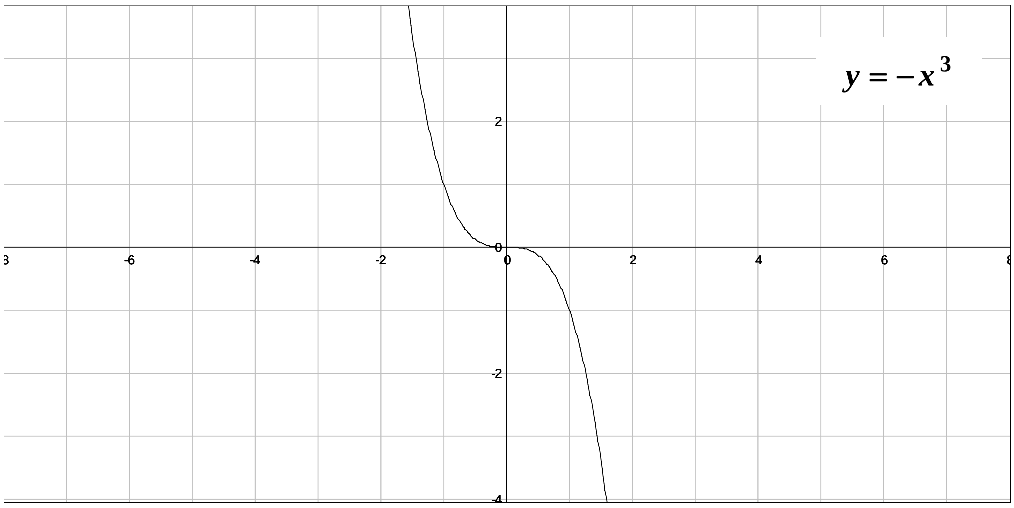


Statement 5

If a function is differentiable and decreasing on (a,b) then its gradient is negative on (a,b).

Counter-example

The function $y = -x^3$ is differentiable and decreasing on R but its gradient is zero at the point x = 0.



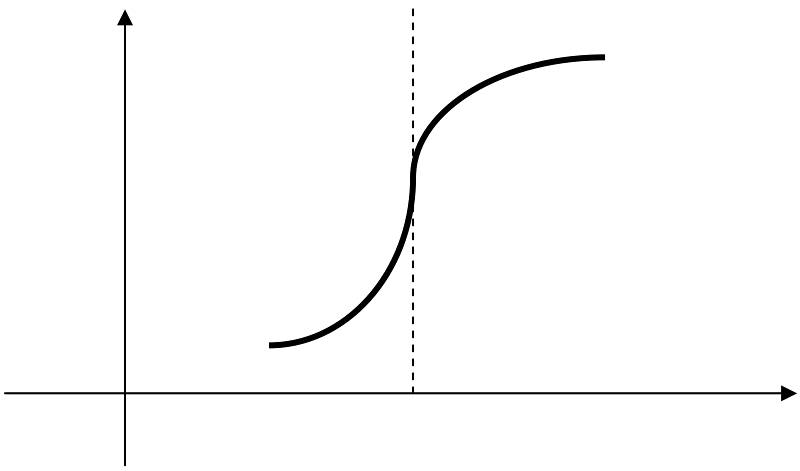
The purpose of this experiment was to see how using counter-examples in class effected students' performance on the test question that required conceptual understanding.

THE RESULTS

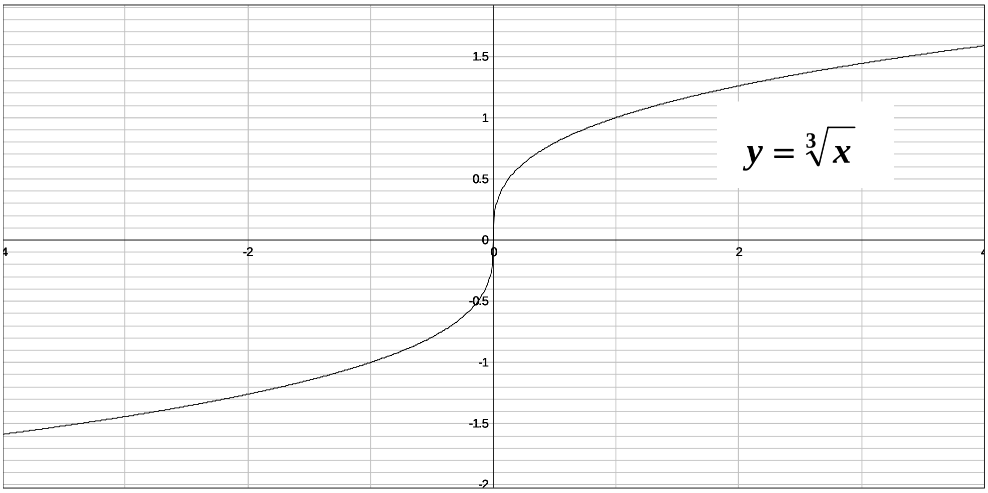
After 8 weeks of study both groups sat the same mid-semester test containing 11 questions: the first 10 questions dealt with skills in techniques, and question 11 tested conceptual understanding.

<u>Question 11</u>. *Sketch a graph of a function that is a continuous and smooth (no sharp corner) at a point but which is not differentiable at that point.*

What we expected from the students was a simple sketch:



Or perhaps something like the cube root function:



The results of the test and question 11 are below:

Group A: Passed the test 13/14=93% Question 11: 11/14=79%
Group B: Passed the test 10/11=91% Question 11: 5/11=45%

DISCUSSION AND CONCLUSION

The students' performance on questions 1-10 was very similar in both groups, as were their overall test results: 93% of the students in group A and 91% in group B passed (received more that 50% of the total marks). When looking at the results of question 11, a significant difference between the two groups is apparent. 79% of the students in group A answered question 11 correctly, versus 45% in group B. This might suggest that group A's conceptual understanding was improved as an immediate result of their work with counter-examples.

As with any case study an essential question is this: to what extent can we generalise these results? Regardless of the answer, employing counter-examples as a pedagogical strategy is certainly worth trying!

There is a well-known book on counter-examples in Calculus: "Counterexamples in Analysis" by B.R.Gelbaum and J.M.H.Olmsted (Holden-Day, Inc., San Francisco, 1964). It is an excellent resource for the teaching and learning of Calculus at an advanced level, but it is well beyond the scope of first-year university Calculus courses, ones that might be based on the popular "Calculus: Concepts and Contexts" by

J. Stewart (Brooks/Cole, Thomson Learning, 2nd ed., 2001) for example. Another supplementary teaching resource is the recently published book “Counter-Examples in Calculus” (Klymchuk, 2004). These two books don’t overlap – all statements and examples are different. The latter book is aimed at filling the niche in the activity on using counter-examples as a pedagogical strategy in teaching/learning of a first-year university Introductory Calculus course.

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